Learning from a toy model: the Kac ring

Scientific models misrepresent their target in that some features of the target are omitted and some others are idealized. These omissions and idealizations are required for mathematical tractability. An important question is therefore how scientists can genuinely learn something from models. Answers to this question have centered on an analysis of scientific models as approximate descriptions of their target. Here models must distort as little as it is compatible with tractability (Bokulich 2008, 2011; Strevens 2008; Weisberg 2007). However such an analysis falls short in accounting for how toy models can teach us things about actual empirical systems — e.g., the Ising model, the Lotka-Volterra model or the Schelling model. Although it may be hypothesized that toy models are approximations of the roughest kind, their degree of approximation is not a sufficient consideration to explain why they are useful in science. Toy models indeed do not aim to describe a particular empirical system for the purpose of providing any particular information about this system. They instead aim to capture general features of a certain class of systems in a way that make information about them easy to infer; this way involves deliberate strong distortions.

In this paper, I contend that toy models are better analyzed in terms of scientific caricatures than in terms of approximate descriptions. What I call scientific caricatures are models that emphasize, even distort, some features of the target and omit others for the purpose of facilitating the inferential work. In other words, scientific caricatures misrepresent their target for users to easily draw information from them. The condition for their success stands nevertheless in that their distortions be harmless: despite their omissions and idealizations, caricatures must still contain a minimal amount of relevant accurate information about the target.

In arguing for such an account of toy models, I develop a case study. I elaborate on the way a toy model is used to study an attempt at explaining the second law of thermodynamics. This model is the Kac ring (Kac 1959). The model exemplifies an irreversible macroscopic behavior in such a way that makes it possible for users to study the Boltzmann's attempt — i.e. the H-theorem — at explaining why macroscopic physical phenomena are irreversible while they result from microscopic phenomena which are reversible (e.g. Boltzmann 1877).

In order to express the H-theorem (1872), Boltzmann studied mechanistically a non-equilibrium ideal gas. He assumed that the gas particles interact by means of repulsive short-range and attractive forces. Thus he obtained an equation which describes the particle and velocity density. Boltzmann used the distribution function in order to introduce a new physical magnitude, which is the function H of the gas. The theorem states that H decreases monotonically over time and reaches a constant value when the particle density in the system equals the density at equilibrium.

Boltzmann also introduced the molecular chaos hypothesis (often called by the German term Stosszahlansatz). In accordance with this hypothesis, the number of collisions between two groups of particles during a certain period dt is proportional to their respective particle density. The hypothesis is highly criticized, in particular because one considers that it provides – rather than explains – the temporal asymmetry of thermodynamical behaviors.

The H-theorem is plagued by two famous paradoxes that lie at the foundations of statistical mechanics: the "reversibility paradox" and the "recurrence paradox":

- the reversibility paradox: If a gas starts from the initial state s_0 and reaches the state St after time t, then, according to the H-theorem, $H_t \leq H_0$. Now if the velocity of every gas particle is reversed, the gas will come back to its initial state st' = s_0 after time t_0 (= t), and H would still continue to decrease so that $H_{t'} = H_0 \leq Ht$. Therefore H cannot decrease over time but remain constant, which contradicts the H-theorem.

- the recurrence paradox: Based on Poincaré's recurrence theorem – which states that a closed dynamical system comes back arbitrarily near its initial state after a sufficiently long period of time – H cannot always decrease over time and is expected to be periodic.

I first present the Kac ring model which was introduced in 1959 by mathematician Mark Kac as an "analog of the classical solution of Boltzmann" to explain macroscopic irreversibility (Kac 1959, p. 99). The Kac ring is a ring with N sites on it. On each site there is a ball which can be either black or white. At each time step, every ball moves counterclockwise to the next site. There is a certain number of fixed actives sites on the ring, which have the following property: when a ball leaves an active site, it switches color, turning white if it was black, black if it was white. I explain why this model is a good representation of the solution of Boltzmann and in particular I show how it expresses the molecular chaos hypothesis.

I second present the lessons that are usually drawn from the model:

First the models accounts for the relaxation phenomenon of a macroscopic system. An isolated system is at equilibrium when all its accessible micro-states are equally likely. Consequently the macro-state which results from the highest number of micro-states is also the most likely. In the Kac ring, the equilibrium obtains when the number of black balls equals the number of white balls.

Second the model illustrates the reversibility and recurrence paradoxes. As I will show, it can find its initial state back and is 2N-periodic.

I then argue that the Kac ring makes it possible to easily draw these lessons precisely because it is a scientific caricature that involves harmless distortions. It distorts the (already idealized) representation of a gas system underlying system, thus highlighting the most relevant aspects of the class of systems being studied.

References

Bokulich, A. (2008). Can Classical Structures Explain Quantum Phenomena? *British Journal for the Philosophy of Science*, 59(2):217–235.

Bokulich, A. (2011). How Scientific Models Can Explain. *Synthese*, 180(1): 33–45.

Boltzmann, L. (1877). Über die Beziehung zwischen dem zweiten Haubtsatze der mechanischen Wärmetheorie und der Wahrscheinlichkeitsrechnung resp. dem Sätzen über das Wärmegleichgewicht Wiener Berichte. In (Boltzmann 1909) Vol. II, paper 42, 76:373–435.

Kac, M. (1959). *Probability and Related Topics in the Physical Sciences.* Interscience Pub., New York.

Strevens, M. (2008). *Depth: An Account of Scientific Explanation*. Cambridge, MA: Harvard University Press.

Weisberg, M. (2007). Three Kinds of Idealization. *Journal of Philosophy*, 104(12):639–659.