Clarifying the Continuous and the Discrete

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January 12, 2018

Long Abstract

Philosophical discussion of the continuous-discrete dichotomy often includes discussion of the so-called analog-digital dichotomy. Some work has gone into showing how these two pairs are actually quite different, and that the second is not even a proper dichotomy (Maley, 2011). Less attention has been paid to the continuous-discrete dichotomy on its own terms, which will be the focus of the present essay. First, I offer some clarity about the nature of the continuous that has been obscured in some of the literature on this topic. Next, I argue that while the continuous-discrete pair truly is dichotomous (in the sense that the terms are mutually exclusive), it is also best to consider the classification of something as either continuous or discrete as relative to an observer, framework, pragmatic purpose, or something along these lines.

Understanding what it is to be discrete is relatively straightforward, even if it may be difficult to define precisely. One intuitive difference sometimes thought to be crucial is that, for those things that are discrete, there is a well-defined "next" thing: words in a sentence, letters in a word, and integers are discrete because for every item there is a preceding and a subsequent item (with exceptions for the first or last items as the case may be). Those things that are continuous are not like this—there is no "next" element—and making continuity clearer illustrates this fact.

Some errors have found their way into the philosophical literature regarding the nature of the continuous. One prominent example is (Goodman, 1968), where being analog/continuous is identified with being infinitely "dense." It may seem tempting to think of continuity in terms of density: if, between any two points you can find infinitely many more, then you have a continuum. Furthermore, for dense sets, there does not seem to be a

well-defined "next" element, a requirement for continuity. For example, we can ask what the next integer after *four* is, but not the next rational number after $1/2$.

Unfortunately, the idea that density is sufficient for continuity is not quite right. To illustrate why, first take a simple example such as the set of rational numbers between 0 and 1. If we choose any two, no matter how close together they are, there is always another between those two. So, if you pick a and b, then between those two is $(a+b)/2$, which we can call c. Now, there is also a point between a and c, and one between c and b. This process can be repeated infinitely, so there are infinitely many points between a and b. Why, then, does this not amount to a continuity?

The simple answer is that continuity requires that there are *no* gaps in the scheme or system in question. Although the rational numbers are dense in Goodman's sense, gaps still exist. Obviously, irrational and transcendental numbers, such as $\sqrt{2}/2$ and π are not rational, and thus there are gaps in the rational numbers. Another way to see that the rational numbers are not continuous is to view them differently, which shows that there *is* a way in which it is sensible to ask of the "next" rational number. Instead of ordering the rational numbers by their value, we can order them by increasing order of the sum of the numerator and denominator:

$$\{0, \tfrac{1}{1}, \tfrac{1}{2}, \tfrac{2}{1}, \tfrac{1}{3}, \tfrac{2}{2}, \tfrac{3}{1}, \tfrac{1}{4}, \tfrac{2}{3}, \tfrac{3}{2}, \tfrac{4}{1}, \tfrac{1}{5}, \ldots\}$$

In this way, for each element there *is* a next rational number. For continuous sets, however, there is no way that they can be ordered such that there are gaps, and thus that there are well-ordered elements (they are not enumerable, in other words). They are "smooth" through-and-through. So, for discrete sets, while it may appear that there is no "next" element in a certain guise or representation of that set, there is a way to enumerate the elements to show that gaps really exist. For continuous sets, this property is entirely lacking.

Having discussed ways in which to clarify what it is to be continuous, the second part of this essay offers an argument that to be continuous or discrete is best viewed as an entirely relative matter. Consider how one might answer the question of whether something is continuous or discrete. At least two ways are available. First, we might ask whether something is continuous or discrete *relative* to a framework, measurement process, one's interests, etc. This is the view I argue for here. Second, we might ask whether something is continuous or discrete in some *absolute*, or general, sense. Let us look at what an answer to the first question might look like. To take a concrete example, we might ask whether a very fine sand or powder is continuous or discrete.

For some purposes, we may want to consider sand or very small particles to be continuous, perhaps like a fluid. If we are measuring time by the amount of sand in an hourglass, or if we are interested in the flow rate of grain through a processing facility, it may be best (for a variety of practical reasons) to treat the sand or grain as a fluid. In other cases, we may care about individual grains of sand. We may be interested in designing materials for a Mars rover that only allow a small number of individual grains to adhere to their surfaces. Cases like these call for us to treat sand as discrete particles.

Now suppose we want to know whether sand is *really* continuous or discrete, and not whether it is best to treat it as one way or another. After all, we may well think that sand is actually discrete particles, even though it can be treated *as if* it were continuous. By what standard could we say that something is *really* continuous or discrete? This is more difficult than it appears at first: individual grains of sand are themselves composed of still smaller particles, which themselves have constituents, etc. The only principled place to look for a final answer would be at the level of fundamental physics. However, this has two consequences, both of which are unsatisfying.

First, this would make all of our common practices regarding whether certain things are continuous or discrete either flawed or inconsequential. If what makes something truly continuous or discrete is a matter of whether fundamental physics ultimately "bottoms out" at discrete particles or continuous fields (for example), then nothing at a higher level makes a difference at all. Fluids, rocks, sand, and everything else would not be continuous or discrete because of anything other than what physics ends up declaring about the fundamental constituents of reality.

Second—and closely related—this would make the distinction between continuity and discreteness a purely hypothetical (and never actual) distinction, at least for physical objects. If we assume (as seems reasonable) that all physical things are ultimately made of the same kind of fundamental constituents, then if those constituents are discrete, then everything will be discrete, and nothing will be continuous. Alternatively, if the fundamental physical constituents are continuous, then everything will be continuous, and nothing will be discrete.

Neither of these objections are incoherent or inconsistent, but they make the distinction between the continuous and the discrete rely on factors irrelevant to why that distinction is made in the first place (i.e. facts about fundamental physics) as well as denying the possibility that there really could be both continuous and discrete phenomena (barring any radically new developments in the kinds of fundamental constituents of physical reality). Pragmatically speaking, it seems best to think of, for example, digital computers are being discrete at the level at which they are, in fact digital: namely, the values of their binary values are such that they can be thought of as being either a "0" or a "1." This is true *even* *though* one might object that these values are *really* continuous, because current flow (and other electrical properties) are themselves continuous. But then again, current flow is *really* discrete, because it is composed of electrons, which are individual entities. But then again…and so on. Better to simply stop at the level we are interested in: digital computers are discrete at a certain level of description.

References

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