Bridging the gap between the continuous and the discrete: the case of absolute continua

Abstract:

It has been claimed that bridging the gap between the domains of discreteness and of continuity, or between arithmetic and geometry, is a central, presumably even the central and the oldest problem in the foundations of mathematics and in the related philosophical fields. Following Russell, it is usually assumed that this problem, along with the problems of infinity and of the infinitesimals are completely solved by way of the so-called *Cantor-Dedekind axiom* which imposes the continuity of the *arithmetic continuum* i.e. the set (or, alternatively, ordered field) $\mathbb{R}$ of real numbers upon the *geometric continuum* – the simple line of Euclidean geometry. During the course of our talk we shall address the question raised by Giulio Vivanti in 1900 concerning the structure of this simplest one-dimensional continuum: *What are the parts into which the line is divided, geometrical points or (infinitesimal) intervals?* The stage is set by contrasting two traditional approaches to answering it, the now standard *Cantorian point-based conception* and the now non-standard (but until Cantor's time dominant) *Aristotelian interval-based conception*. It is shown how Cantor's explicit negation of one of the central tenets of Aristotelianism as regards the structure of continua – the so-called *Zeno's axiom* and the condition of *dimensional homogeneity* which it introduces – brought about the arithmetico-set-theoretic pointillist conception of the real line which has become the basis of all the standard formulations of analysis, analytic and synthetic theories of the geometric linear continuum and of all the axiomatic theories of continuous magnitudes and the scientific study of continuous phenomena more generally.

More importantly, bearing in mind the Archimedean nature of $\mathbb{R}$, acceptance of the Cantor-Dedekind axiom as the bridge between continuity and discreteness (lines and points or numbers) renders infinitesimals entirely redundant for the analysis of the structure of the geometric continuum. That is to say, by accepting mainstream Cantorian pointillism we automatically discard not only Aristotelian intervalism, but also an entire cluster of alternative theories of continua, namely non-Archimedean ones. However, there is more to continuity than what is entailed by mere

archimedicity. We shall illustrate how the uncritical espousal of the *Cantor-Dedekind academic dogma* helped instill a certain prejudice against infinitesimals among contemporary mathematicians, philosophers and even historians of mathematics which resulted in many superficial and even falsified accounts of 19th century infinitesimal mathematics and a peculiar *damnatio memoriae* of the pioneering works of Paul du Bois-Reymond, Otto Stolz and especially Giuseppe Veronese to whose contributions the central part of the talk is devoted. Namely, we shall present Veronese's direct challenge to the Cantor-Dedekind axiom by reconstructing his views on the foundations of geometry and, more specifically, the way he introduces "a new species of continua", non-Archimedean i.e. infinitesimal-enriched ones.

Veronese's conception will be interpreted as a call for rehabilitation of the classical Greek method and a viable yet forgotten alternative to Cantorian pointillism in the sense that it could (and should) be regarded as a non-Archimedean variation and an expansion of Aristotelian intervalism by way of a logically consistent introduction of infinitesimal intervals as constituents of the linear continuum. In line with contemporary neo-Aristotelian philosophy of mathematics, we will show that Veronese's theory is indeed a *par excellence* example of Aristotelianism for its emphasis upon the perceptual i.e. visual model in explaining the ontological status and the mutual relations of geometrical objects. Then, starting from the intuitive Aristotelian interval-based continuum, we shall see how Veronese's construction of a linearly ordered magnitude-system whose elements are bounded segments of the Euclidian line turns out to be a construction of an *absolute continuum* i.e. a maximally inclusive non-Archimedean ordered field which contains both the Cantor-Dedekind and the Aristotelian continuum as parts, thus making them merely *relative continua*.

From the geometrical point of view (which for Veronese is the relevant and primary point of view when it comes to the question concerning the structure of continua), this means that the line contains infinitesimal and non-infinitesimal intervals as well as points, although in different senses of "contain". Namely, a difference is drawn between "being composed of" and "being covered by" and it is shown that even though the line is covered everywhere by points, it still may not be composed of points, unless something is added – bounded infinitesimal line segments which conjoin the points bounding them, thus making them continuous. Finally, we present Veronese's hilbertian observations regarding the use of the "language of points" and the "language of intervals" in geometry in connection with some contemporary axiomatic attempts at overcoming the "great struggle" between Cantorians and Aristotelians. All of this will allow us to see how the struggle is easily overcome once we operate with Veronese-style absolute continua – the two theories are reconciled since they are only trivially different which is to say that intrageometrically they are equivalent since both are relative continua which can be easily isolated within the richer structure of Veronese's absolute continuum by means of the Archimedean property.

Turning from geometry to arithmetic, we shall show how one can still keep an analogue of the Cantor-Dedekind axiom even in a non-Archimedean setting by constructing novel, extended number systems, richer and denser than the standard reals. In other words, it is absolute geometric and arithmetic continua and not the Cantor-Dedekind continuum which bridge the gap between geometry and arithmetic in an entirely satisfactory – absolute – way. We shall present a variety of absolute arithmetic continua which may be used for such a purpose, starting from the so-called *Veronese numbers*, followed by Levi-Civita's *numeri monosemii*, Robinson's *hyperreal* and finally Conway's *surreal numbers*. By means of special microscopic and telescopic techniques, we shall see in which way standard and non-standard geometrical and arithmetical objects are related.

In the concluding remarks we pledge for a reconsideration of infinitesimalism by sketching out some of the prospects of working with absolute continua by uncovering a whole range of implications of studying the structures introduced by Veronese and those who followed in his way, interesting for contemporary philosophy of mathematics, philosophy of physics and even mathematics and physics proper.

Key words: absolute and relative continua, point-based and interval-based continua, geometric and arithmetic continua, infinitesimals, Giuseppe Veronese.